Entrance Lengths for Transport with a Specified Surface Flux

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In this note an analysis is made of the length required for temperature or concentration profiles to reach a fully developed shape for the case of heat or mass transfer inside circular tubes having a uniform surface flux. The method proposed has the considerable advantage of avoiding the necessity for integration of the partial differential equation describing transport in the entrance region. Instead, the method utilizes the downstream fully developed transport profiles which can be determined with relative ease.

Based on a definition of entrance length as the position downstream from the start of heating where the fully developed temperature profile first applies, an overall energy balance determines a rigorous lower bound for the entrance length. A hypothesized qualitative boundary layer model of the transport development process leads to the conclusion that the lower bound referred to above will ordinarily be nearly achieved as an equality in practice. Comparison of the new approach with certain previous calculations bears this out. The present analysis results in an equation that is analogous to the Lyon (1951) integral for the developed Nusselt number under constant flux conditions and requires the same information and mathematical techniques for its evaluation.

Although other geometries and specified flux conditions are amenable to analysis by the methods given here, in this note we treat only the important case of transport in a circular tube with a constant surface flux. For this case, the following equation describes the dimensionless entrance length:

$$\frac{L}{D} \geqq \frac{1}{u_b R^3} \int_{r=0}^{R} ur \int_{s=0}^{r} \frac{1}{s(\alpha + \epsilon_H)} \int_{t=0}^{s} utdtdsdr \quad (1)$$

This expression is evaluated here for certain situations in both laminar and turbulent flow. For the laminar flow of a power law non-Newtonian fluid in a circular tube, Equation (1) becomes

$$\frac{L}{D} \geqq \frac{Pe}{4} \left\{ \frac{A}{4} \left(1 - \frac{1}{(1+b)^2} \right) - \frac{1}{Nu} \right\} \quad (2)$$

The appropriate expression for the Nusselt number is given in the Supplement.* For $n = 0$ and $n = 1$, corresponding to slug and parabolic flow respectively, the prediction of Equation (2) agrees well with earlier detailed calculations.

For turbulent flow, an eddy diffusivity distribution is required for the evaluation of Equation (1). In this case, by using a distribution due to Reichardt (1951) and assuming that the turbulent Prandtl number is constant, a simple approximate formula can be derived. This formula, which is valid for all molecular Prandtl numbers of interest, is as follows:

$$\frac{L}{D} \geqq \frac{Pe}{32} \left(\frac{1}{1 + 0.91\beta} \right) \quad (3)$$

This formula also seems to agree reasonably well with certain earlier results, except for large Prandtl numbers.

The principal significance of this work is the demonstration that in specified flux situations transport entrance length information can be determined without a detailed knowledge of the transport in the entrance region. This concept can be used for various geometries and for laminar or turbulent flows. It is also applicable to situations where the developed velocity and eddy diffusivity profiles might be represented experimentally or empirically.

THE MODEL

The physical case considered is as follows. The fluid enters the heated section of the pipe with a uniform temperature θ_0 and with a fully developed velocity profile. In addition, it is assumed that there is no viscous dissipation or axial diffusion.

An energy balance over the length of pipe from the entrance of the heated section to the point at which the temperature profile first becomes fully developed gives an expression for the thermal entrance length as

$$L = \frac{\rho C_p}{q_w R} \int_0^r ur\,(\theta - \theta_0)\,dr \quad (4)$$

If we are heating, for example, the initial temperature, θ_0 must be less than or equal to θ_c, and this inequality yields

$$L \geqq \frac{\rho C_p}{q_w R} \int_0^r ur\,(\theta - \theta_c)\,dr \quad (5)$$

where θ_c is the centerline temperature at the first point downstream where the fully developed profile applies. That this inequality should be nearly an equality in practice is suggested by the following argument. If we imagine the development of the thermal profile to take place in the form of a thermal boundary layer that grows from the wall toward the centerline, then we would expect the thermal entrance length to be slightly downstream of the point where the thermal boundary layer completely fills the tube. This corresponds to the last point where the centerline temperature is equal to the initial fluid temperature. The preceding conjecture will be borne out in a later comparison of results. For conditions of constant physical properties, negligible viscous dissipation of energy and no axial transport, the energy equation may be expressed as follows:

$$u \frac{\partial \theta}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left[r(\alpha + \epsilon_H) \frac{\partial \theta}{\partial r} \right] \quad (6)$$

The fully developed temperature profile corresponds to $\partial\theta/\partial x$ = constant, and for this situation Equation (6) can be solved to yield

$$\theta - \theta_c = \frac{2q_w}{\rho C_p u_b R} \int_{s=0}^{r} \frac{1}{s(\alpha + \epsilon_H)} \int_{t=0}^{s} utdtds \quad (7)$$

An analogous equation for the concentration difference for the case of mass transfer may be derived subject to the

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assumptions of constant physical properties, no axial diffusion and no diffusion-induced normal velocity. Equation (7) for the temperature difference $\theta - \theta_c$ may be substituted into Equation (5) to give an expression for the thermal entrance length.

$$\frac{L}{D} \geq \frac{1}{u_b R^3} \int_{r=0}^{R} ur \int_{s=0}^{r} \frac{1}{s(\alpha + \epsilon_H)} \int_{t=0}^{s} utdtdsdr \quad (8)$$

As mentioned previously, the method used to derive the entrance length formula is not restricted to the circular tube geometry. Integrals expressing the entrance length may be derived for other geometries such as concentric annuli, parallel plates, etc. The calculations become somewhat more difficult in cases where it is necessary to determine the position of the extremum of the developed temperature profile. This technique could also be applied to situations where the velocity and eddy diffusivity profiles are not known but can be measured, as would be the case for heat transfer in a packed bed.

In the next sections analytical representations of the integrals in Equation (8) are derived for certain specific cases.

LAMINAR NEWTONIAN AND POWER-LAW NON-NEWTONIAN FLOW

For laminar Newtonian flow, substitution of the parabolic velocity profile in Equation (8) yields $L/D \geq (7/192)\mathrm{Pe}$. This result may be compared to the calculations of Seigel et al. (1958) who used the method of separation of variables and Sturm-Liouville theory to solve the energy equation for the boundary condition of a uniform wall heat flux. The results obtained by these authors predict that the local Nusselt number has approached to within 5% of its fully developed value in an entrance length given by $L/D = .0428\ Pe$.

Thus it is seen that for practical purposes inequality (8) is nearly an equality in this case, which was conjectured in earlier discussion. It should be noted that, strictly speaking, the fully developed profile is only an asymptotic result which applies better and better arbitrarily far downstream. The usual determination of entrance length involves a specification of the Nusselt number as some fraction of its asymptotic value. This means that the entrance length is at the discretion of the specifier and is therefore somewhat ambiguous. The approach given here has the advantage that no ambiguity of the previous type arises. The price paid for this is that we must pretend that the fully developed profile begins to apply exactly at some finite point.

In the case of power-law non-Newtonian flow, the entrance length relation is given by Equation (2). Further details for this case are given in the Supplement.*

EFFECT OF AXIAL CONDUCTION

The customary asymptotic downstream constant flux solution to the energy equation also applies when axial conduction is present. This effect, which is important primarily for liquid metals, greatly complicates matters as regards determination of the entrance length in the classical manner. However, the effect of axial conduction is easily incorporated into the approach given here. When axial conduction is to be considered, the overall energy balance includes the section preceding the start of heating, since due to axial conduction the effect of heating propagates upstream. For this situation the only change that occurs in Equation (8) is that the term $\frac{q_a}{4q_w}$ which accounts for axial conduction at the downstream section, is added to the right-hand side. For laminar flow in a tube this correction term is $-1/Pe$. The effect of axial conduction in laminar flow is thus seen to be significant for Peclet numbers less than about 20.

TURBULENT FLOW WITH $\sigma = 1$

Just as the Lyon integral reduces to the Reynolds analogy for $\sigma = 1$ under appropriate conditions, the entrance length formula, Equation (8), derived here can also be greatly simplified under the same conditions. That is, for $\sigma = 1$, if we take $\epsilon_H = \epsilon_m$ and use $u = u_b$ in the proper locations, then the entrance length integral can be determined without specifying any particular eddy diffusivity distribution.

To derive this result we proceed as follws. Take $u = u_b$ in the inside integral of Equation (8). This yields

$$\frac{L}{D} \geq \frac{1}{u_b^2 f R^2} \int_0^R ur\,(u_c - u)\,dr \quad (9)$$

where is has been noted that $\int_0^r sds/(\nu + \epsilon_M)$ can be expressed in terms of u_c and u by using $\int_0^r = \int_0^R - \int_r^R$ together with the integral of the momentum equation. If we now take $u = u_b$ in the factor outside the bracket in Equation (9), the final result is

$$\frac{L}{D} \geq \frac{1}{2f}\left(\frac{u_c}{u_b} - 1\right) \quad (10)$$

To implement Equation (10), it is necessary to specify a velocity distribution in order to calculate f and u_c. For turbulent flow the logarithmic profile $u = C_1 \ln y + C_2$ is useful except very close to the wall. This profile, with appropriate values of C_1 and C_2, has been used to derive a relation between friction factor and Reynolds number which is widely used, namely, $1/\sqrt{f} = 4.06 \log (Re\sqrt{f}) - 0.6$. For simplicity and consistency, this profile is used here to determine u_c/u_b. In this way we find that $u_c/u_b = 1 + 3.75\sqrt{f}/2$. When this is substituted into (10), we finally get

$$\frac{L}{D} \geq \frac{1.3258}{\sqrt{f}} \quad (11)$$

From the friction factor relationship $1/\sqrt{f} = 4.06 \log (Re\sqrt{f}) - 0.6$, it can be established that in the limit of large Reynolds number, $1/\sqrt{f}$ varies as 4.06 log Re, and therefore $L/D \to \infty$ logarithmically with Re as $Re \to \infty$. Equation (11) agrees well with existing data and with other calculations, as will be discussed in the next section. Since the logarithmic velocity profile was used in Equation (10) to calculate u_c and f, the corresponding results in Equation (11) should extrapolate correctly to very large Reynolds numbers.

TURBULENT FLOW WITH ARBITRARY σ

In the previous section for $\sigma = 1$ and $\epsilon_m = \epsilon_H$, it was not necessary to specify an eddy diffusivity distribution in order to evaluate the entrance length. For $\sigma \neq 1$, such a specification is necessary. The distribution chosen for our purposes as the best compromise between simiplicity and accuracy is that due to Reichardt (1951). The Reichardt diffusivity distribution can be written as $\epsilon_M{}^+ = \frac{kR^+}{6}$

* See footnote on page 867.

$[1 + \bar{r}^2 - 2\bar{r}^4]$. This expression agrees very well with experimental measurements in the central region of a circular tube. It is somewhat less accurate near the wall, but this would be expected to be unimportant except possibly at large Prandtl numbers. However, as shown in the Supplement, the results are insensitive to the variation of the eddy diffusivity near the wall even at large Prandtl numbers.

The evaluation of Equation (8) using the Reichardt diffusivity distribution involves calculations whose details appear in the supplement. This includes some interesting asymptotic analysis which leads to a considerable simplification of the final results. The final formula for the entrance length that applies for any Prandtl number is

$$\frac{L}{D} \geq \frac{\sigma Re}{32}\left[1/(1 + .91\beta)\right] \qquad (12)$$

where $\beta \equiv \sigma\,\sigma_T\,k\,Re\,\sqrt{f/2}/12$. It is interesting to note that the entrance length L/D predicted by Equation (12) for $\sigma = 1$ is rather close to that of Equation (11) that is, 21.4 versus 19.6 at $Re = 10^5$).

The predicted thermal entrance length of Equation (12) may be compared to the calculations of Sparrow et al. (1957) who solved the energy equation in the thermal entrance region for particular values of the Prandtl and Reynolds numbers. For $\sigma = 0.7$ and for $Re = 10^5$, these authors give a thermal entrance length based on $Nu/Nu_\infty = 1.05$ of $L/D = 13$. For the same conditions Equation (12) predicts $L/D = 21.3$. Sparrow et al. used the eddy diffusivity expression of Deissler (1955) in their calculations, and this may account for some of the discrepancy in this comparison. The calculated entrance length of Sparrow et al. is very sensitive to the value of Nu/Nu_∞ used to define the entrance length. On the other hand, the definition adopted here is fixed once and for all and yields a lower bound for the entrance length so defined.

DISCUSSION OF RESULTS

The approach taken in this paper to the problem of transport entrance lengths for specified situations has the advantages of simplicity and nonambiguity. The nature of the calculation as well as comparisons with alternative approaches show that the idealized concept of entrance length used here has a practical validity. The great simplicity is based on being able to make entrance length predictions without the need for detailed transport knowledge in the entrance region.

Although in this paper detailed results were given only for transport in a circular tube, the method can obviously be extended to other geometries. The method also allows various additional effects, such as viscous dissipation, for example, to be taken into account.

In general, the entrance length integral developed here would require numerical techniques for its evaluation in the case of turbulent flow. In this paper, however, approximate formulas were obtained by making some reasonable simplifying assumptions. Agreement of the results of the entrance length calculations performed here with those given elsewhere are good with one notable exception. Recently Notter and Sleicher (1972) have solved the turbulent Graetz problem with what they consider to be a very good eddy diffusivity distribution. Their entrance lengths are considerably smaller and show a maximum at fixed Reynolds numbers for increasing σ. No such behavior is indicated by Equation (8), even when the eddy diffusivity of Notter and Sleicher is used. The differences between these results may possibly be attributed to the different definitions of the entrance length used. The customary definition of entrance length, as the length required for the Nusselt number to fall to a specified fraction of the asymptotic value, may be very sensitive to Prandtl number at large Prandtl numbers.

NOTATION

A = function of flow behavior index, $A = (3n + 1)/(n + 1)$
b = function of flow behavior index, $b = (n + 1)/(2n)$
C_1, C_2 = constants for logarithmic velocity distribution
D = tube diameter
f = friction factor, $f = 2\tau_w/\rho u_b^2$
k = constant in Reichardt eddy diffusivity expression, $k = 0.4$
K = power-law parameter, consistency index
C_p = constant pressure heat capacity
L = transport entrance length
n = power-law parameter, flow behavior index
Nu = Nusselt number
Pe = Peclet number, $Pe = u_b D/\alpha$
q_a = axial heat conduction flux
q_w = wall heat flux
r = radial coordinate
$\bar{r}$ = dimensionless radial coordinate, $r = r/R$
R = tube radius
R^+ = dimensionless tube radius, $R^+ = Ru^*/\nu$
Re = Reynolds number, $Re = Du_b/\nu$
s,t = dummy variables for r
u = velocity
u_b = bulk velocity
u_c = centerline velocity
u^* = friction velocity, $u^* = u_b\sqrt{f/2}$
x = axial coordinate
y = distance from wall

Greek Letters

α = thermal diffusivity
β = $0.4\,\sigma\sigma_T Re\,\sqrt{f/2}/12$
ϵ_H = eddy diffusivity for heat
$\epsilon_H{}^+$ = dimensionless eddy diffusivity, $\epsilon_H{}^+ = \epsilon_H/\nu$
ϵ_M = eddy diffusivity for momentum
θ = temperature
θ_c = centerline temperature
θ_0 = inlet temperature
ν = kinematic viscosity
ρ = fluid density
σ = Prandtl or Schmidt number
σ_T = turbulent Prandtl number, $\sigma_T = \epsilon_M/\epsilon_H$
τ = shear stress
τ_w = shear stress at the wall

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Gas Mixture Adsorption on Molecular Sieves

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The note of Myers (1973) recently published in the Journal attributes a "serious mistake . . based upon an unjustifiable procedure" as being responsible for "incorrect *conclusions*" in the paper of Danner and Wenzel (1969). In this latter paper the authors evaluated the ideal adsorbed solution theory (Myers and Prausnitz, 1965) as a practical method for predicting a priori binary adsorption data from the pure gas isotherms. In contrast to Myers' comments, it is claimed here that the procedure followed was both reasonable and justified and that no mistake was made in the calculations or conclusions.

Use of the Myers and Prausnitz method for predicting gas mixture adsorption data requires calculation of the spreading pressure of each pure adsorbate as a function of pressure. Accurate calculation of spreading pressures requires low surface coverage data. For strongly adsorbing systems, this in turn requires such extremely low pressures that they generally cannot be measured in standard adsorption equipment. If one cannot obtain such data, some means of extrapolating the data to low coverage is necessary if only pure gas data are to be used to predict the mixture adsorption behavior. Danner and Wenzel made the extrapolation to zero coverage by using the Langmuir equation, a procedure Myers has characterized as unjustified. The selection of this equation, however, was based on the relative accuracy of the Langmuir, *n*-layers BET, and Freundlich equations in representing the lowest portion of the experimental isotherms of oxygen, nitrogen, and carbon monoxide on molecular sieves.

Myers suggests that the solution to the low coverage extrapolation problem is the application of the Gibbs adsorption isotherm to binary mixtures at constant pressure as described by Van Ness (1971). There are two serious deficiencies in his suggestion. First, the application of this method defeats the primary goal of having a means of predicting the mixture data from the pure gas isotherms. The calculation now requires an integral of the experimental binary adsorption data over the entire composition range to derive a correction factor (integration constant). Thus, although Myers claims that this modification of the theory shows "that the theory of ideal solution predicts . . adsorption on molecular sieves"; the method, in fact, no longer predicts mixture results from pure gas data but instead simply correlates the experimental mixture data.

Second, the correlating method described by Myers still retains the deficiency of requiring the extrapolation of the isotherm data to zero coverage for at least one of the adsorbates. The calculated correction factor is a difference between the spreading pressures of the two pure gases; and in order to calculate absolute values of the spreading pressures, one of the actual spreading pressure curves must be determined. In the example given by Myers, the curve for nitrogen was calculated from the data of Danner and Wenzel. Although these nitrogen data extended down to a coverage of one-fourth the monolayer coverage rather than only to one-half coverage as in the case of carbon monoxide, this is still not a sufficiently low range of coverage to allow extrapolation with confidence. Although Myers has been quite critical of Danner and Wenzel's method of making this extrapolation, he does not report how he carried out the calculations for nitrogen. He must have relied, however, on some similar type of procedure for nitrogen that he has termed a serious mistake when used by Danner and Wenzel for the case of carbon monoxide.

In summary, although the note of Myers does suggest that the ideal adsorbed solution theory may be a useful correlating method for molecular sieve systems, the suggested modification does not allow its use as an a priori predictive method and does not remove the necessity for extrapolating the pure component data. The treatment of the data reported by Danner and Wenzel was a rational and justified attempt to compare the theory of Myers and Prausnitz with the other methods of predicting binary adsorption data using only the pure gas isotherms.

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